Last time: - Hilbert spaces

- $\ell^2(\mathbb{Z}) = \widehat{\bigoplus_{n\in\mathbb{Z}}} \mathbb{C}$

Today: (A very general) measure theory

also called a "ring" as in "ring of sets"

Def: A subtractive system is

a set S a binary operation $+$

and a partially defined binary operation $\setminus$ subtraction

and an element $0 \in S$, satisfying

(Associativity) $a+(b+c) = (a+b)+c$

(Commutativity) $a+b = b+a$

(Identity) $a+0 = a$

(Add/Subtract) $(a+b)\setminus a$ is defined and $\quad a + [(a+b)\setminus a] = a+b$

(Additivity) $\quad [(a+b)\setminus b] + [(b+c)\setminus b] = (a+b+c)\setminus b$

((Cancellation)
or (Excision)) $\quad [(a+b+c)\setminus b] \setminus [(b+c)\setminus b]$

$$= (a+b+c)\setminus(b+c)$$

(Subtractive identity) $\quad a\setminus a = 0$

$$(a+b)\setminus[(a+b)\setminus a] = a$$

We write $(S, +, \setminus)$ for this data.

Elements $a, b \in S$ such that $(a+b)\setminus a = b$

Then we say that a and b are disjoint, or orthogonal, denoted $a \perp b$

not a symmetric relation!

E.g. $(\mathbb{R}, +, -)$ or $(\mathbb{C}, +, -)$ → any two elements are disjoint

$(\mathbb{R}_{\geq 0}, +, -)$

$(\mathbb{R}_{\geq 0} \cup \{\infty\}, +, -)$ → $(\infty + b) - \infty = 0 \;\; \forall b$

So $\infty \perp 0$

but $\infty \not\perp b$ if $b \neq 0$.

$(\mathcal{P}(X), \cup, \setminus)$ where X is a set and $\mathcal{P}(X)$ is its power set

only defined when $A \supseteq B$!

$X = \{1,2,3\}$ $A = \{1,2\}$ $B = \{2,3\}$, then

$(A \cup B) \setminus A = \{3\} \neq B$.

$(G, +, -)$ for any abelian group G,

in particular a vector space

Def: Let $\mathcal{H}$ be a Hilbert space.

The lattice of closed subspaces of H

is the set

$$L(\mathcal{H}) = \{ W \subseteq \mathcal{H} \mid W \text{ is a closed subspace} \}$$

Define $A \hat{+} B = \overline{A + B}$

and $A \setminus B = A \cap B^{\perp}$.

Then $(L(H), \hat{+}, \setminus)$ is a subtractive system.

A sequence (a_n) in a subtractive system is said to be pairwise disjoint if $a_n \perp a_m$ whenever $n \neq m$.

, σ-ring

Def: A summable system or σ-system is a subtractive system S along with a collection D_S of sequences in S, called summable sequences, along with a map $\sum : D_S \to S$, satisfying

1) if $(a_n)_{n=1}^{\infty}$ is summable, then so is $(a_n)_{n=N}^{\infty}$, and

$$\sum (a_n)_{n=1}^{\infty} \setminus \left(\sum_{n=1}^{N-1} a_n \right) = \sum (a_n)_{n=N}^{\infty}$$

2) if a_0 is disjoint to every element of $(a_n)_{n=1}^{\infty}$ which is summable, then $(a_n)_{n=0}^{\infty}$ is summable and

$$a_0 + \sum (a_n)_{n=1}^{\infty} = \sum (a_n)_{n=0}^{\infty}$$

If every pairwise disjoint sequence is summable, we say that S is a Dynkin system or d-system

E.g. $(\mathbb{R}, +, -)$ or $(\mathbb{C}, +, -)$ or $(\mathbb{R}_{\geq 0}, +, -)$

with $D_S = \{$all sequences st $\sum_{n=0}^{\infty} a_n$ converges$\}$

or $D_S = \{$all sequences st $\sum_{n=0}^{\infty} a_n$ abs. converges$\}$

If V is any Banach space, then

$(V, +, -)$ with $D_S = \{ \cdot \; \sum_{n=0}^{\infty} \|a_n\| < \infty \}$

$(P(X), \cup, \setminus)$ with $D_S = \{$all pairwise disjoint seq.$\}$

$(L(\mathcal{H}), \hat{+}, \setminus)$ with D_S "

$(\mathbb{R}_{\geq 0} \cup \{\infty\}, +, -)$ with $D_s = \{\text{all sequences}\}$

Dynkin systems

Def: If S_1 is a Dynkin system and S_2 is a summable system, then a d-map from S_1 to S_2 is a function $\alpha : S_1 \to S_2$ such that

1) $\alpha(0) = 0$
2) $\alpha((a+b) \setminus b) = \alpha(a+b) \setminus \alpha(b)$
3) If (a_n) is summable then so is $(\alpha(a_n))$, and
$$\alpha\left(\sum(a_n)\right) = \sum(\alpha(a_n))$$

Note α only preserves sums of disjoint elements in general!

We say a subset S' of a Dynkin System S is a sub-d-System if $0 \in S'$ and S' is closed under $\setminus$ and Σ when they are defined. Not necessarily under $+$!

A sub-d-system of $\mathcal{P}(X)$ which contains X and is closed under finite intersections is called a σ-algebra on X.

d-maps from $\mathcal{P}([n])$ to $L^2([m])$ for $n \leq m$ are equivalent to a choice of n pairwise orthogonal subspaces $V_1, \dots, V_n$ in $L^2([m])$. We send $S \in \mathcal{P}([n])$ to $\sum_{i \in S} V_i$.

In general a d-map from a σ-algebra to $L(\mathcal{H})$ for $\mathcal{H}$ a Hilbert space gives a subspace-valued measure, which allow us to define measurements, direct integrals, and the spectral theorem in infinite-dimensional Hilbert space.

A d-map from a σ-algebra to $\mathbb{R}_{\geq 0} \cup \{\infty\}$ is called a measure, which allow us to define integration. Similarly we can define signed measures, complex measures, and vector measures, using d-maps from a σ-algebra to $\mathbb{R}$, $\mathbb{C}$, or V.

A d-map between two σ-algebras is called a measurable function

A nonzero $\mathbb{d}$-map from $L(\mathcal{H})$ to $\mathbb{R}_{\geq 0}$

is called a Gleason measure

which is like a probablility measure,

but for events which are subspaces of $\mathcal{H}$.

$\rightsquigarrow$ Lets us derive Born's rule using

Gleason's Thm